Lemma 1. *In the setting of 2 $\xrightarrow{p}$ 1 QRAC with NS option, if Alice encodes NS bit with a state(e.g., (0,0)) and the other three states into the qubit, the maximal expected success probability is* $\frac{5+\sqrt{5}}{8}$.

Proof. Suppose the four states are ρ_{00}, ρ_{01}, ρ_{10}, and ρ_{11}, corresponding to Alice sending information 00, 01, 10 and 11 respectively. Without loss of generality, encode ρ_{00} with $|NS\rangle$. The other three states ρ_i can be represented as $\frac{\mathcal{I}+\vec{r_i}\cdot\vec{\sigma}}{2}$, where $\vec{r_i}$ is the Bloch representation satisfying $\|\vec{r_i}\| \leq 1$. Each orthogonal measurements $\{M_s^t\}_{t=0,1}$ has the form $M_s^0 = \frac{\mathcal{I}+\vec{m}_s^0\cdot\vec{\sigma}}{2}$ and $M_s^1 = \frac{\mathcal{I}\vec{m}_s^1\cdot\vec{\sigma}}{2}$, whose Bloch vector are antipodes.

The success probability is

$$\begin{aligned} P_s &= 1/4 + 1/8 \sum_{i\in\{01,10,11\},s\in\{0,1\}} Tr[\rho_i M_s^{i_s}] \\ &= 1/4 + 1/8 \sum_{i\in\{01,10,11\},s\in\{0,1\}} (\frac{1+\vec{r_i}\cdot\vec{m}_s^{i_s}}{2}) \end{aligned} \tag{1}$$

The 1/4 term after the first equality is account for the case where Alice sends $|NS\rangle$ and Bob decode it as (0,0) successfully; the second equality holds because in 2-dimension, for two states ρ, M with Bloch vector $\vec{r}, vecm$, $Tr[\rho M] = 1/2(1+\vec{\rho}\cdot\vec{m})$.

We know that $P_s \geq 0.5$ since any optimal strategy should perform no less than random guess for any i. Therefore, $\sum_{s\in\{0,1\}}((-1)^{i_s}\vec{r_i}\cdot\vec{m}_s) \geq 0$ holds. If $\exists i$, $\|\vec{r_i}\| <1$, then scale it to a unit vector in Bloch sphere would give a better result, meaning that the optimal solution is the one with pure state encoding scheme.

Consider the plane through origin in the Bloch ball which contains both measurements, depicted in Figure 1.$|t\rangle_s$ represents measuring bit s and get result t$\in \{0,1\}$. Notice that $|0\rangle_s$ and $|1\rangle_s$ are in the same diameter of the circle. Denote $\theta_{is^t} \in [0,\pi]$ as the angle between $\vec{r_i}$ and $\vec{m_s^t}$. The summation in equation 1 is proportion to $cos^2(\theta_{is^t}/2)$, which will increase as the angle θ_{is^t} decreases. When projecting each state ρ_i into the plane and re-scale it to the circumference of the circle, the inner product $\vec{r_i}\cdot\vec{m_s^{i_s}}$ will increase. Therefore, when it is an optimal solution, the three states should lie in the same plane as the orthogonal measurement, as drawn in Figure 1.

Using geometry relations, equation 1 becomes

$$
\begin{aligned}
P_s &\leq \max_{\alpha,\beta_1,\beta_2,\gamma} {}^1\!/_4 + {}^{(cos^2\beta_1+sin^2(\alpha+\beta_1))}\!/_8 + {}^{(cos^2\beta_2+sin^2(\alpha+\beta_2))}\!/_8 + {}^{(cos^2\gamma+cos^2(\alpha-\gamma))}\!/_8 \\
&= {}^1\!/_4 + {}^1\!/_4 \max_{\alpha,\gamma}\{\max_{\beta} cos^2\beta + sin^2(\alpha+\beta) + {}^1\!/_2(cos^2\gamma + cos^2(\alpha-\gamma))\} \\
&= {}^1\!/_2 + {}^1\!/_4 \max_{\alpha,\gamma}\{\max_{\beta} sin(\alpha)sin(\alpha+2\beta) + {}^1\!/_2(cos^2\gamma + cos^2(\alpha-\gamma))\} \\
&= {}^1\!/_2 + {}^1\!/_4 \max_{\alpha,\gamma}\{sin(\alpha) + {}^1\!/_2(cos^2\gamma + cos^2(\alpha-\gamma))\} \\
&= {}^1\!/_2 + {}^1\!/_4 \max_{\alpha}\{sin(\alpha) + (cos^2(\alpha/2))\} = {}^5\!/_8 + {}^1\!/_8 \max_{\alpha}\{2sin(\alpha) + (cos(\alpha))\} = {}^{5+\sqrt{5}}\!/_8
\end{aligned} \tag{2}
$$

Where α, β and γ are defined in Figure 1. The maximal value can be reached when $\alpha = arctan({}^1\!/_2)$, $\beta_1 = \beta_2 = {}^{\pi-2\alpha}\!/_4$, and $\gamma = {}^{\alpha}\!/_2$. Notice that there exists mirror symmetry in this solution with respect to the bisector of angle 2α.

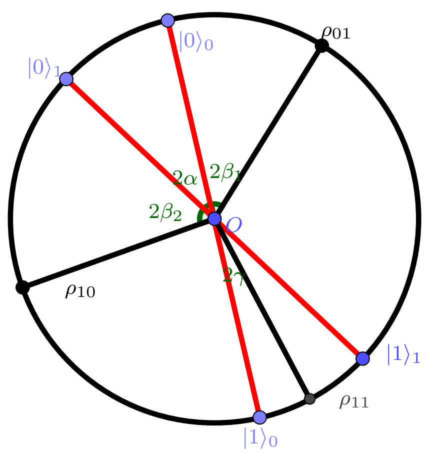

Figure 1: Three states and two measurements in a Bloch Sphere

□

Lemma 2. *In 2 $\xrightarrow{p}$ 1 QRAC with NS option, the maximum success probability is* $\frac{5+\sqrt{5}}{8}$.

Proof. From Lemma 1, the success probability of $\frac{5+\sqrt{5}}{8}$ is achievable.
Define each state $\rho_i = q_i + (1-q_i)\rho_i$, where q_i is the probability that Alice choose not to send the qubit when she wants to send information $i \in \{0,1\}^2$. When the qubit is sent and measurement $\{M_j^t\}_{t=0,1}$ is performed to get bit

j, denote the success retrieval probability as $p_{ij} = Tr[\rho_i M_j^{i_j}]$. Suppose $r_j \in [0, 1]$ is the probability of decoding bit j as 0 when Bob does not receive anything. The total probability of success is

$$\begin{aligned} P_s &= 1/8 \sum_i (p_{i1} + p_{i2})(1 - q_i) + (q_{00} + q_{01})r_1 \\ &\quad + (q_{10} + q_{11})(1 - r_1) + (q_{00} + q_{10})r_2 + (q_{01} + q_{11})(1 - r_2) \\ &\leq 1/8 \sum_i (p_{i1} + p_{i2})(1 - q_i) + \max\{q_{00} + q_{01}, q_{10} + q_{11}\} \\ &\quad + \max\{q_{00} + q_{10}, q_{01} + q_{11}\} \end{aligned} \tag{3}$$

For each i, P_s is a linear function with respect to q_i, therefore a global maximum solution should satisfy $q_i \in 0, 1$. Without loss of generality, suppose $\max\{q_{00} + q_{01}, q_{10} + q_{11}\} + \max\{q_{00} + q_{10}, q_{01} + q_{11}\} = 2q_{00} + q_{01} + q_{10}$. Clearly, $\forall i, j,\ p_{ij} \leq 1, \Rightarrow p_{11} + p12 \leq 2 \Rightarrow \frac{\partial P_s}{\partial q_{00}} \geq 0$, hence $q_{00} = 1$. Also, $\frac{\partial P_s}{\partial q_{11}} \leq 0, \Rightarrow q_{11} = 0$.
If for i=(0,1) or (1,0), $\frac{\partial P_s}{\partial q_i} \leq 0$, then clearly, $(q_{00}, q_{01}, q_{10}, q_{11}$ is a global maximum, where only one state is encoded with $|NS\rangle$. Otherwise, if $\exists i \in \{(0, 1), (1, 0)\}$, $\frac{\partial P_s}{\partial q_i} > 0$, P_s becomes maximum when $q_i = 1$ the success probability of retrieving one bit from information i is $p_i = 1/2[(p_{i1}+p_{i2})(1-q_i)+q_i] \leq 0.5$. The total probability of success $p_s \leq 1 - 0.5/4 = 0.875 < \frac{5+\sqrt{5}}{8}$.

□

Lemma 3. *In $2 \xrightarrow{p} 1$ QRAC with NS option, if denote Alice's qubit states as $|\psi_i\rangle$ and Bob needs to detect the r_{th} bit with POVM $\{\Lambda_r^j\}_{j=0,1}$, $i \in \{(0, 1)^2\}$ and $j \in \{(0, 1)\}$, $\max_{\{\langle\psi_i|\},\{\Lambda_r\}}(\min_{i,r} Tr[\Lambda_r^{i_r}|\psi_i\rangle\langle\psi_i|]) \leq \frac{2+\sqrt{2}}{4}$, i.e, the S/NS degree of freedom cannot improve this attribute.*

Proof. We can adopt similar methods used in Lemma 1 and Lemma 2. Following arguments from Lemma 1, we can find a plane in the Bloch sphere where all the states and measurements live in. The orthogonal measurements are shown in Figure 2 and the respective states $\{\rho_i\}$ when Alice chooses to send the qubit. Denote $q_i \in [0, 1]$ as the possibility that Alice sends the qubit for information i, and let $r_j \in [0, 1]$ be the probabilities that Bob depict the j_{th} bit as 0 when he receives nothing. Refer to Figure 2 for the definitions of the angles in the following discussion.
Assume the contrary that for each information and each measurement, the

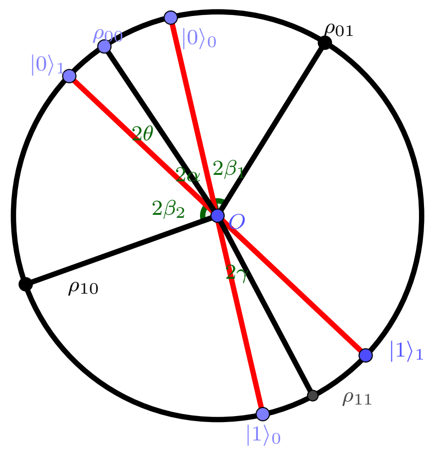

Figure 2: Four states and two measurements in a Bloch Sphere

probability of retrieval is greater than $p_s := \frac{2+\sqrt{2}}{4}$. Without loss of generality, suppose $\alpha \leq \pi/2$(otherwise we can flip measurement and re-index the states, which should get the same result). Notice that

$$p_{01} = \min\{q_{01}cos^2\beta_1 + (1-q_{01})r_1, q_{01}sin^2(\alpha+\beta_1) + (1-q_{01})(1-r_2)\} \quad (4)$$

$$p_{10} = \min\{q_{10}cos^2\beta_2 + (1-q_{10})(1-r_1), q_{10}sin^2(\alpha+\beta_2) + (1-q_{10})r_2\} \quad (5)$$

If $\beta_1, \beta_2 \geq \pi/8$, since $p_{01}, p_{10} \geq p_s$, but $cos^2\beta_1, cos^2\beta_2 \leq p_s$, $\Rightarrow r_1, 1-r_1 > p_s$. contradiction. Similar argument can be applied when $\beta_1, \beta_2 \leq \pi/8$. Without loss of generality, suppose $\beta_1 \geq \pi/8$ and $\beta_2 \leq \pi/8$. Therefore, $r_1, r_2 > p_s$. Let

$$g_1(\alpha, r_1, 1-r_2) = \max_{q_{01},\beta_1} p_{01} \quad (6)$$

$$g_2(\alpha, r_2, 1-r_1) = \max_{q_{10},\beta_2} p_{10} \quad (7)$$

We know that $g_1 = g_2 = g(\alpha, a, b)$ which satisfies $\frac{\partial g}{\partial a} \geq 0, \frac{\partial g}{\partial b} \geq 0$. Suppose further that $r_1 \geq r_2$. If at lease one of the partial differential is zero(e.g.$\frac{\partial g}{\partial b} \geq 0$), then we can adjust $r_2 \Rightarrow r_1$ and $\min\{g_1, g_2\}$does not decrease. Otherwise, if $r_1 > r_2 \Rightarrow g_1 > g_2$, $p \leq \min\{g_1, g_2\} = g_2$. Adjusting r_1 and r_2 to make them closer, and g_2 will increase. When $r_1 = r_2$, g_2 takes the maximum value. In this case,

$$p_s < g_2 \leq \max_{q_{10},\beta_2,\alpha} {}^1\!/_2(q_{10}cos^2\beta_2 + (1-q_{10})(1-r_1) + q_{10}sin^2(\alpha+\beta_2) + (1-q_{10})r_2)$$
$$= \max_{q_{10},\beta_2,\alpha} q_{10}(cos^2\beta_2 + sin^2(\alpha+\beta_2)) + 1 - q_{10}$$
$$\Rightarrow q_{10}(sin(\alpha+2\beta_2)sin\alpha) > \sqrt{2}/2 \Rightarrow q_{10} > \frac{1}{\sqrt{2}sin\alpha} \geq 1. \tag{8}$$

Contradiction! Therefore it is not possible to increase the minimum success probability with the NS option.

□

Remark. In $2 \xrightarrow{p} 1$ QRAC with a qu-trit substituting qu-bit as the resource, the maximal success probability is still $\frac{5+\sqrt{5}}{8}$, i.e., the same as using qubit with NS option, although the qu-trit is a strictly stronger resource.

It is known that for n≫1, there exists an n $\xrightarrow{p}$ 1 QRAC with expected success probability $p_q(n) \approx \frac{1}{2} + \sqrt{\frac{2}{3\pi n}}$ by random measurements. Also, in the classical case, the success probability $p_c(n) \approx \frac{1}{2} + \sqrt{\frac{1}{2\pi n}}$. Such probability can be achieved using strategy with identity decoding function and majority encoding function. These two facts are used for a lower bound for n $\xrightarrow{p}$ 1 QRAC with NS option.

Theorem 4. *(Lower bound)For every n≫1 there exists an n $\xrightarrow{p}$ 1 QRAC strategy with NS option, which has expected success probability p(n)* $\approx \frac{1}{2} + \sqrt{\frac{7}{6\pi n}}$

Proof. Consider the following strategy. For a given c∈[0,1], let $n_q = n - \lfloor n(1-c) \rfloor$ and $n_c = \lfloor n(1-c) \rfloor$. Alice prepare the qubit to encode information for her last n_q bits according to $n_q \xrightarrow{p} 1$ QRAC case. then encode the first n_c bits classically using the majority encoding scheme: when there is more 1 than 0 in the first n_c bits, Alice send the qubit; if they are of the same frequency, send the qubit with probability of 0.5; do not send otherwise. Bob guess bit j of Alice's information according to Table 1 below.

Next we derive the expected success probability P_s. When n is large, $n_c \approx nc$ and $n_q \approx n(1-c)$. The probability of send or not sent should be equal because of the majority code scheme. Hence,

$$p(n) = P(j \in [1, nc]p_c(nc) + P(j \in [nc+1, n])(\frac{1}{2}P(NS) + p_q(n(1-c))P(S))$$
$$\approx c(\frac{1}{2} + \frac{1}{\sqrt{2\pi nc}}) + (1-c)(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2}(\frac{1}{2} + \sqrt{\frac{2}{3\pi n(1-c)}}))$$
$$= \frac{1}{2} + (\sqrt{\frac{c}{2\pi}} + \sqrt{\frac{2(1-c)}{3\pi}})\frac{1}{\sqrt{n}} \tag{9}$$

Let c=arccos($\sqrt{\frac{3}{7}}$), equation 9 becomes p(n) $\approx \frac{1}{2} + \sqrt{\frac{7}{6\pi n}}$, which is an asymptotic lower bound of the QRAC with NS option.

	Qubit Sent	Qubit Not Sent
bit j$\in [1, n_c]$ required	guess 1	guess 0
bit j$\in [n_c + 1, n]$ required	measurement	guess at random

Table 1: Bob's strategy

□