LSH formulation for SU(3) Lattice Gauge Theory

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1 Introduction

Path integral formulation is often used in solving the SU(3) gauge theory. Given a SU(3) gauge-invariant Lagrangian $\mathcal{L}$, the partition function is[12]

$$Z = \int D\psi e^{-iS[\psi]} \tag{1}$$

where the action $S[\psi] = \int d^4x \mathcal{L}[\psi, \partial\psi]$. If the Lagrangian is real, then after a Wick rotation $t \to -it$, the action S is purely imaginary and the metric becomes Euclidean. We can perform Monte-Carlo method to sample the set of configuration of field ψ and the probability density assigned to each configuration is well-defined.

However, sign problem would emerge when we consider topological terms[7]. For example, if we perform a chiral rotation, after a change in the path-integral method, there is a term in the Lagrangian that looks like $m\bar{\psi}e^{i\gamma_5}\psi$, which makes the Lagrangian complex. The partition function 1 is now an path integral with oscillating phase factor and the usual probability interpretation doesn't work.

Therefore, we adopt a Hamiltonian formulism to work non-pertubatively. We discretize the space and adopt Kogut-Susskind formulation[1], where staggered fermion is used to reduce the degeneracy. The operators corresponding to interaction term involves non-local gauge group elements and introduce non-abelian gauge constraints. To satisfy those constraints, we split the gauge link into two components, each involves only local gauge transformations[3]. This is called a Quantum Link Model(QLM).

In SU(2) gauge theory, the separation reduces to two separate SU(2) representation with the same weight, which can be implemented by angular momentum representation(i.e., $\exists$ raising and lowering operators $J^{\pm}$ and the J_3 operator)[3]. Generally, the algebra suggests that a Schwinger Boson formulation can be adopted to satisfy the commutation relation for any semisimple Lie group, especially SU(3).

In this way, the gauge field is generated by a set of bosonic operators[2, 4, 5], which live in some representation of the relevant Lie group. These operators, combined with the fermionic operators, can form Gauge-nvariant local operators which combine as the Hamiltonian. This process is called Loop, String, Hadron formalism(LSH)[8, 9] and it can completely tern non-abelian Gauge constraints to a set of abelian Gauss constrains. Relevant Quantum computation algorithms can be designed and implemented based on these operators to calculate relevant properties. Such form is studied in the case of SU(2) symmetry[8], and a pure gauge theory has been developed[9].

In the following sections, I will try to extend the LSH formalism to the full SU(3) gauge theroy, with the presence of both fermionic field and gauge field.

2 Preliminaries[10–12]

For a gauge field $A = A^a T^a$, where T^a are the generators of the Lie algebra, define covariant derivative, $D_\mu \psi^i = \partial_\mu \psi^i - i A_\mu^a [T^a(R)]^i_j \psi^j$.

2.1 SU(3) Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \Sigma_q \bar{\psi}^q_i (i\gamma^\mu (D_\mu)_{ij} - m)\psi^q_j \tag{2}$$

Where each ψ^a is a Dirac spinor. The variation with respect to $\bar{\phi}$ shows that the equation of the field is

$$i\frac{d\psi}{dt} = A_\mu(x)\frac{dx}{dt} \tag{3}$$

This shows that we can view A as the connection over the principle bundle where ψ lives. The curvature is defined as $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, where f^{abc} are the structural constants.

The geometrical interpretation of A field motivates us to define a parallel transport, aka a Wilson line

$$U[x_i, x_f; C] = \mathcal{P}exp(i\int_{x_i}^{x_f} A) \tag{4}$$

where $\mathcal{P}$ is the path ordering and C is a specific path with x_i initial point and x_f end point. It is nontrivial since SU(3) is non-Abelian.

2.2 Gauge transformation

For fermionic field,

$$\psi \to \Lambda[R]\psi \tag{5}$$

where $\Lambda[R] \in SU(3)$ is a gauge symmetry in representation R. For simplicity, sometimes we drop R if the representation is clear under the context.

For gauge field,

$$\begin{aligned} A_\mu \to& \Lambda A_\mu \Lambda^{-1} + \Lambda \partial_\mu (\Lambda^{-1}) \\ F_{\mu\nu} \to& \Lambda F_{\mu_n u} \Lambda^{-1} \end{aligned} \tag{6}$$

Under such transformation, a covariant derivative transforms as a fermionic field,

$$D_\mu \psi \to \Omega D_\mu \psi \tag{7}$$

and a Wilson line transforms as

$$U[x_i, x_f; C] \to \Lambda(x_i)U[x_i, x_f; C]\Lambda^\dagger(x_f) \tag{8}$$

The SU(3) Gauss Law reads

$$\begin{aligned}
&\Sigma_i(E^a_L(n,i) + E^a_R(n,i))|phys\rangle = 0 \\
&\Sigma_i[(E^a_L(n,i) + E^a_R(n-i,i)) + \frac{1-(-)^n}{2} - (\psi^{a\dagger}_n\psi^a_n - \psi^{a\dagger}_n\psi^a_n)]|phys\rangle = 0
\end{aligned} \tag{9}$$

This defines the physical space, $\mathcal{H}_{phys} \subset \mathcal{H}_{gauge}$.

2.3 QCD Hamiltonian

We can select Weyl Gauge $A^0 = 0$ and get the Hamiltonian as[11]

$$H = \int d^dx[-i\bar{\psi}\gamma^i D_i\psi + m\bar{\psi}\psi + 1/2(E^2 + B^2)] \tag{10}$$

where d is the dimension of consideration, $B^a_k = \epsilon_{klm}(\partial_l A^a_m - g/2 f^{abc}A^b_l A^c_m)$.

We can quantize the R^d space and get the Kogut-Susskind Hamiltonian[1],

$$H = H_I + H_M + H_E + H_\Box \tag{11}$$

with each term corresponds to a term in equation 10

$$\begin{aligned}
H_I &= -t\Sigma_{x,i}(\psi^\dagger_x U_{x,i}\psi_{x+i} + h.c.) \\
H_M &= m\Sigma_x(-)^x\psi^\dagger_x\psi_x \\
H_E &= g_e^2/2\Sigma_{x,i}E^2_{x,i} \\
H_B &= g_m^2/2\Sigma_{x;i\neq j\in\{1,2,\ldots d\}}(\Box_{x,i,j}) + h.c.
\end{aligned} \tag{12}$$

where the plaquettes $\Box_{x,i,j} = U_{x,i}U_{x+i,j}U^\dagger_{x+j,i}U^\dagger_{x,j}$ with U the wilson lines.

The first term is the hopping term, which discribes the interaction of a fermionic field with its corresponding gauge field. It can be clearly interpreted that fields at different locations interact via the parallel transporter U.

Quantizing the space will introduce a momentum bound to the problem. There will be a Brillouin Zone in the momentum space, where the corners should all be identified. This will introduce redundancy in the description of the problem, known as the fermion doubling problem. Here we introduce a staggered fermion method to reduce such redundancy by diagonalizing the interaction term in the Hamiltonian, so each entry in the Dirac spinor is separated from other entries and the equation is identical to all 4 entries. WLOG, we can then consider only the 1st entry of the Dirac spinors. Mathematically we can introduce a transformation $\psi \to \psi T(x)$, where $T(x) = \gamma_1^{x_1}\gamma_2^{x_1}\gamma_3^{x_3}$ in 3 dimensional space. This will diagonalize the interaction term and introduce a new $(-)^x := (-1)^{x_1+x_2+x_3}$ factor in the mass term.

In the Weyl gauge, what are left in the Hamiltonian are the colorelectric and colormagnetic field, which is the familiar electric field and magnetic field in U(1) gauge theory. The colorelectric term discretises intuitively, whereas the colormagnetic term is less obvious. We aim to preserve the SU(3) gauge symmetry in the discretised formula, and the natural object to find is the Wilson line $U[x_i, x_f,]$, which is the only "observable" gauge field operator since all local details between two adjacent sites should be lost during the discretization. From equation 8, the only gauge-invariant obect to consider is the Wilson loop

$$W[C] = tr\mathcal{P}exp(i \oint A) \tag{13}$$

Since the space is discretised, the smallest such loop is a plaquett $\square_{x,i,j}$. A bigger loop can be construct by multiplying the small plaquettes together. All inner lines will be cancelled out since $U[x_i, x_f, C] = U[x_f, x_i, -C]^{-1}$ and what survives is the outer edges forming the bigger loop.

2.4 Quantum Link Model

Since $U[x_i, x_f; C] \to \Lambda(x_i)U[x_i, x_f; C]\Lambda^\dagger(x_f)$, U belogs to two different local Gauge symmetry groups. Hence we can split $U = U_L U_R$ and hopefully make each half-link transform locally under two gauge symmetries respectively, with

$$\begin{aligned} U_L &\to \Lambda(x_i)U_L \\ U_R &\to U_R\Lambda(x_i) \end{aligned} \tag{14}$$

We can find the generators of such transforms, $E_{L/R}$, satisfying[6, 9]

$$\begin{aligned} [E_L^a(n,i), U(n,i)] &= T^a U(n,i) \\ [E_R^a(n,i), U(n,i)] &= U(n,i)T^a \end{aligned} \tag{15}$$

such transformation indicates that E are the colour-electric field operators in the Weyl Gauge (since $E^i = \dot{A}^i$ which satisfy the same commutative relations).

The Casimir constraint is

$$E_L^a E_L^a = E_R^a E_R^a \tag{16}$$

at each link.

3 Irreducible Shwinger Boson operators in SU(3)

We can reformulate equation 15 using Schwinger Boson formulation.[2, 9]

$$\begin{aligned} E_L^a &= a_L^\dagger T^a a_L - b_L T^a b_L^\dagger \\ E_R^a &= a_R^\dagger T^a a_R - b_R T^a b_R^\dagger \end{aligned} \tag{17}$$

where each a, b is a triplet define at respective link at location (n,i). The commutation relation 15 turns to

$$
\begin{aligned}
&[a, a] = 0 = [b, b] \\
&[a_l^\alpha, a_{l'\beta}^\dagger] = \delta_\beta^\alpha \delta_{ll'} \\
&[b_{l\alpha}, b_{l'}^{\dagger\beta}] = \delta_\alpha^\beta \delta_{ll'}
\end{aligned}
\tag{18}
$$

and all commutators at different link vanishes. Such process is called prepotential formulation. We can see that

$$
\begin{aligned}
a^\alpha &\rightarrow \Lambda_\beta^\alpha a^\beta \\
b_\alpha &\rightarrow b_\beta \Lambda_\alpha^\beta = b_\beta [(\Lambda)^\dagger]_\beta^\alpha
\end{aligned}
\tag{19}
$$

we can thus identify a as living in representation 3 and b living in representation $\bar{3}$, the fundamental and antifundamental representations of SU(3) respectively.

However, such representation of the gauge field generating a space $\mathcal{H}_{prepotential} \supset \mathcal{H}_{gauge}$, which has additional $U(1) \times U(1) \times Sp(2, R) \times Sp(2, R)$ symmetry[9]. The $U(1) \times U(1)$ symmetry requires

$$
N(R) = M(L), N(L) = M(R) \tag{20}
$$

where the operators are defined in section 5.3. The additional symplectic symmetry motivates us to define the color neutural(i.e. gauge invariant) Sp(2,R) operators as[9]

$$
\begin{aligned}
k_-(l) &\equiv a(l) \cdot b(l) \\
k_+(l) &\equiv a^\dagger(l) \cdot b^\dagger(l) \\
k_0(l) &\equiv \frac{1}{2}(a^\dagger(l) \cdot a(l) + b^\dagger(l) \cdot b(l) + 3)
\end{aligned}
\tag{21}
$$

with $l \in \{L, R\}$. The Hilbert spaces generated by prepotential operators can be decomposed into a direct sums of subspaces, each with a spin associated to and is a copy of the gauge Hilbert space. Therefore we can select a "spin neutral" subspace as our gauge space, in which we redefine the prepotentials as[9]

$$
\begin{aligned}
A_\alpha^\dagger(l) &= a_\alpha^\dagger(l) - F_l k_+(l) b_\alpha(l) \\
B^{\dagger\alpha}(l) &= b^{\dagger\alpha}(l) - F_l k_+(l) a^\alpha(l)
\end{aligned}
\tag{22}
$$

with $l \in \{L, R\}$, $F_l = \frac{1}{N(l)+M(l)+1}$. Notice that they transform as $a_\alpha^\dagger$ and $b^{\dagger\alpha}$ respectively.

The gauge link operator is

$$
U_\beta^\alpha = B^{\dagger\alpha}(L)\eta A_\beta^\dagger(R) + A^\alpha(L)\theta B_\beta(R) + (B(L) \wedge A^\dagger(L))^\alpha \delta(A(R) \wedge B^\dagger(R))_\beta \tag{23}
$$

where

$$
\begin{aligned}
&\eta = \eta_L \eta_R, \theta = \theta_L \theta_R, \delta = \delta_L \delta_R \\
&\eta_L = \frac{1}{\sqrt{B(L) \cdot B^\dagger(L)}} \\
&\theta_L = \frac{1}{\sqrt{A^\dagger(L) \cdot A(L)}} \\
&\delta_L = \frac{1}{\sqrt{A(L) \wedge B^\dagger(L) \cdot (B(L) \wedge A^\dagger(L))}} \\
&\eta_R = \frac{1}{\sqrt{A^\dagger(R) \cdot A(R)}} \\
&\theta_R = \frac{1}{\sqrt{B(R) \cdot B^\dagger(R)}} \\
&\delta_R = \frac{1}{\sqrt{A(R) \wedge B^\dagger(R) \cdot (B(R) \wedge A^\dagger(R))}}
\end{aligned}
\tag{24}
$$

The gauge link operator U can be split into two parts, each transforms locally with the gauge symmetry associated to the site as promised in section 2.4. We have[9]

$$
\begin{aligned}
U_L[3] &= \begin{pmatrix} B^{\dagger 1}(L)\eta_L & A^1(L)\theta_L & (B(L) \wedge A^\dagger(L))^1 \delta_L \\ B^{\dagger 2}(L)\eta_L & A^2(L)\theta_L & (B(L) \wedge A^\dagger(L))^2 \delta_L \\ B^{\dagger 3}(L)\eta_L & A^3(L)\theta_L & (B(L) \wedge A^\dagger(L))^3 \delta_L \end{pmatrix} \to \Lambda[3] U_L[3] \\
U_R[3] &= \begin{pmatrix} A^1(R)\eta_R & B^{\dagger 1}(R)\theta_R & (B(R) \wedge A^\dagger(R))^1 \delta_R \\ A^2(R)\eta_R & B^{\dagger 2}(R)\theta_R & (B(R) \wedge A^\dagger(R))^2 \delta_R \\ A^3(R)\eta_R & B^{\dagger 3}(R)\theta_R & (B(R) \wedge A^\dagger(R))^3 \delta_R \end{pmatrix} \to U_R[3] \Lambda[3]^\dagger
\end{aligned}
\tag{25}
$$

4 Coupling with fermionic field

4.1 alternative way of writing the Schwinger boson field

For the same excitation of gauge field, there are two ways of writing the half-link operators, each transform as representation 3 or $\bar{3}$. This can be seen as a natural isomorphism(unlike the case in SU(2), this is complex conjugation, NOT linear transformation) between two representations. Notice the symmetry between A and B in equation 23. Define $\bar{\eta}_l, \bar{\theta} = l$ and $\bar{\delta}_l$ as equation 24, with all $A \leftrightarrow B$. We can do the same exchange of operators in equation 25 and get

$$
\begin{aligned}
U_L[\bar{3}] &= \begin{pmatrix} B_1(L)\bar{\eta}_L & A_1^\dagger(L)\bar{\theta}_L & (B(L)^\dagger \wedge A(L))_1 \bar{\delta}_L \\ B_2(L)\bar{\eta}_L & A_2^\dagger(L)\bar{\theta}_L & (B(L)^\dagger \wedge A(L))_2 \bar{\delta}_L \\ B_3(L)\bar{\eta}_L & A_3^\dagger(L)\bar{\theta}_L & (B(L)^\dagger \wedge A(L))_3 \bar{\delta}_L \end{pmatrix} \to \Lambda[\bar{3}] U_L[\bar{3}] \\
U_R[\bar{3}] &= \begin{pmatrix} A_1^\dagger(R)\bar{\eta}_R & B_1(R)\bar{\theta}_R & (B(R)^\dagger \wedge A(R))_1 \bar{\delta}_R \\ A_2^\dagger(R)\bar{\eta}_R & B_2(R)\bar{\theta}_R & (B(R)^\dagger \wedge A(R))_2 \bar{\delta}_R \\ A_3^\dagger(R)\bar{\eta}_R & B_3(R)\bar{\theta}_R & (B(R)^\dagger \wedge A(R))_3 \bar{\delta}_R \end{pmatrix} \to U_R[\bar{3}] \Lambda[\bar{3}]^\dagger
\end{aligned}
\tag{26}
$$

4.2 Complete Kogut-Susskind Hamiltonian

Notation. distinguish ψ_a and ϕ^b. They represent quark and antiquark fields respectively, each transformed as 3 or $\bar{3}$.

Equation 12 can be rewritten in locally gauge invariant operators in the following form

$$\begin{aligned} H_I &= -t\Sigma_{x,i}(\psi_x^\dagger U_{L;x,i}[3])(U_{R;x,i}[3]\psi_{x+i} + \phi_x^\dagger U_{L;x,i}[\bar{3}])(U_{R;x,i}[\bar{3}]\phi_{x+i}) + h.c. \\ H_M &= m\Sigma_x(-)^x(\psi_x^\dagger\psi_x - \phi_x^\dagger\phi_x \\ H_E &= g_e^2/2\Sigma_{x,i}E_{x,i}^2 \\ H_B &= g_m^2/2\Sigma_{x;i\neq j\in\{1,2,\dots d\}}(\Box_{x,i,j}) + h.c. \end{aligned} \tag{27}$$

All can be formulated using the locally gauge invariant operators defined in the next section.

5 SU(3) LSH operators

5.1 Pure Gauge Loop

Singlets made from gauge field. Invariant under SU(3). Bilinear form.

$$\begin{aligned} L_{ab}^{++} &= a^\dagger(R)_\alpha b^\dagger(L)^\alpha \\ L_{ab}^{--} &= a(R)^\alpha b(L)_\alpha = (L_{ab}^{++})^\dagger \\ L_{ba}^{++} &= b(R)_\alpha a(L)^\alpha \\ L_{ba}^{--} &= b^\dagger(R)^\alpha a^\dagger(L)_\alpha = (L_{ba}^{++})^\dagger \\ L_b^{+-} &= b^\dagger(R)^\alpha b(L)_\alpha \\ L_b^{-+} &= b(R)_\alpha b^\dagger(L)^\alpha = (L_b^{+-})^\dagger \\ L_a^{+-} &= a^\dagger(R)_\alpha a(L)^\alpha \\ L_a^{-+} &= a(R)^\alpha a^\dagger(L)_\alpha = (L_a^{+-})^\dagger \end{aligned} \tag{28}$$

Notice $L_b^{+-} = -L_a^{+-}$ and $L_{ab}^{++} = -L_{ba}^{++}$ in SU(2), since fundamentals and antifundamentals are linearly dependent(in fact, isomorphic) in SU(2). This is not the case in SU(3).

Adopting virtual site formulation to avoid Mandelstam constraints, the following loop operators only appear in those sites with 3 edges.

$$\begin{aligned} L_{ij}^{a+b+} &= a^\dagger(i)\cdot b^\dagger(j) \\ L_{ij}^{a-b-} &= a(i)\cdot b(j) \\ L_{ij}^{a+a-} &= a^\dagger(i)\cdot a(j) \\ L_{ij}^{a-a+} &= a(i)\cdot a^\dagger(j)(L_{ij}^{a+a-})^\dagger \\ L_{ij}^{b+b-} &= b^\dagger(i)\cdot b(j) \\ L_{ij}^{b-b+} &= b(i)\cdot b^\dagger(j) = (L_{ij}^{b+b-})^\dagger \end{aligned} \tag{29}$$

where $i, j \in \{1, 2, 3...2d\}$ denote the direction of edges. Again, there is 1 more pair of operators because the independence of a&b.

There is also a new kind of creation/annihilation operators at virtual sites, genuine to SU(3) in multidimensional:

$$\begin{aligned} A^{\dagger}_{ijk} &= \epsilon^{\alpha\beta\gamma} a^{\dagger}_{\alpha}(i) a^{\dagger}_{\beta}(j) a^{\dagger}_{\gamma}(k) \\ A_{ijk} &= \epsilon_{\alpha\beta\gamma} a^{\alpha}(i) a^{\beta}(j) a^{\gamma}(k) \\ B^{\dagger}_{ijk} &= \epsilon_{\alpha\beta\gamma} b^{\dagger,\alpha}(i) b^{\dagger,\beta}(j) b^{\dagger,\gamma}(k) \\ B^{\dagger}_{ijk} &= \epsilon^{\alpha\beta\gamma} b_{\alpha}(i) b_{\beta}(j) b_{\gamma}(k) \end{aligned} \tag{30}$$

5.2 String

Gauge field interacting with fermionic field. Bilinear form. Notice that more operators than SU(2) at the presense of antiquark fields and anti-gluon fields.

$$\begin{aligned} S^{b,3+}_{in} &= b(R)_{\alpha} \psi^{\dagger\alpha} \\ S^{b,3-}_{in} &= b^{\dagger}(R)^{\alpha} \psi_{\alpha} = (S^{b,3+}_{in})^{\dagger} \\ S^{a,3-}_{in} &= a^{\dagger}(R)_{\alpha} \psi^{\alpha} \\ S^{a,3+}_{in} &= a(R)^{\alpha} \psi^{\dagger\alpha} = (S^{a,3-}_{in})^{\dagger} \\ S^{b,\bar{3}+}_{in} &= b^{\dagger}(R)^{\alpha} \phi^{\dagger}_{\alpha} \\ S^{b,\bar{3}-}_{in} &= b(R)_{\alpha} \phi^{\alpha} = (S^{b,\bar{3}+}_{in})^{\dagger} \\ S^{a,\bar{3}-}_{in} &= a(R)^{\alpha} \phi_{\alpha} \\ S^{a,\bar{3}+}_{in} &= a^{\dagger}(R)_{\alpha} \phi^{\dagger}_{\alpha} = (S^{a,\bar{3}-}_{in})^{\dagger} \end{aligned} \tag{31}$$

$$\begin{aligned} S^{b,3+}_{out} &= b(L)_{\alpha} \psi^{\dagger\alpha} \\ S^{b,3-}_{out} &= b^{\dagger}(L)^{\alpha} \psi_{\alpha} = (S^{b,3+}_{out})^{\dagger} \\ S^{a,3-}_{out} &= a^{\dagger}(L)_{\alpha} \psi^{\alpha} \\ S^{a,3+}_{out} &= a(L)^{\alpha} \psi^{\dagger\alpha} = (S^{a,3-}_{out})^{\dagger} \\ S^{b,\bar{3}+}_{out} &= b^{\dagger}(L)^{\alpha} \phi^{\dagger}_{\alpha} \\ S^{b,\bar{3}-}_{out} &= b(L)_{\alpha} \phi^{\alpha} = (S^{b,\bar{3}+}_{out})^{\dagger} \\ S^{a,\bar{3}-}_{out} &= a(L)^{\alpha} \phi_{\alpha} \\ S^{a,\bar{3}+}_{out} &= a^{\dagger}(L)_{\alpha} \phi^{\dagger}_{\alpha} = (S^{a,\bar{3}-}_{out})^{\dagger} \end{aligned} \tag{32}$$

5.3 Number operators

Gauge flux operators

$$\begin{aligned}N_L &= a^\dagger(L)\cdot a(L)\\N_R &= a^\dagger(R)\cdot a(R)\\M_L &= b^\dagger(L)\cdot b(L)\\M_R &= b^\dagger(R)\cdot b(R)\end{aligned}\tag{33}$$

Quark number operators

$$\begin{aligned}N_\psi &= \psi^{\dagger a}\psi_a\\N_\phi &= \phi_a^\dagger\phi^a\end{aligned}\tag{34}$$

5.4 Hadron operators

Need to be invariant under SU(3). We can use upper and lower index to track the quark/antiquark operators, which live in 3 (fundamental)/$\bar{3}$(antifundamental) representation respectively. Gauge invariant requirements indicate that only η_b^a,ϵ^{abc} and ϵ_{abc}

Mesons: bilinear form, made up of a quark and its antiquark.

$$\begin{aligned}H^{++} &= -\frac{1}{2!}\psi^{\dagger\alpha}\phi_\beta^\dagger\eta_\alpha^\beta\\H^{--} &= \frac{1}{2!}\psi_\alpha\phi^\beta\eta_\beta^\alpha = (H^{++})^\dagger\end{aligned}\tag{35}$$

The other type of operators are $H^{a-} = \psi_\alpha\psi^{\dagger,\alpha}$ and H^{b-}, which are related to $H^{a+} = \psi^{\dagger,\alpha\psi_\alpha}$ and H^{b+} by anticommulation relation and it only involves a constant before annihilated by vacuum, hence can be ignored.

Baryons: not appearing in SU(2) doublets since no rank (0,3) antisymmetric tensor. Corresponds to 3 quarks and 3 antiquarks. This seems not showing up in the final Hamiltonian since the degree is more than 2.

$$\begin{aligned}H^{3+} &= \frac{1}{3!}\psi^{\dagger\alpha}\psi^{\dagger\beta}\psi^{\dagger\gamma}\epsilon_{\alpha\beta\gamma}\\H^{3-} &= \frac{1}{3!}\psi_\alpha\psi_\beta\psi_\gamma\epsilon^{\alpha\beta\gamma}\\H^{\bar{3}+} &= \frac{1}{3!}\phi_\alpha^\dagger\phi_\beta^\dagger\phi_\gamma^\dagger\epsilon^{\alpha\beta\gamma}\\H^{\bar{3}-} &= \frac{1}{3!}\phi^\alpha\phi^\beta\phi^\gamma\epsilon_{\alpha\beta\gamma}\end{aligned}\tag{36}$$

6 Further Work

Commutation relation of the operators can be calculated and relevant quantum simulation algorithms can be developed based on the new operators. The advantage of this formulation is that gauge redundancy is turned into Abelian Gauss laws, which is more computationally efficient.

In $n \geq 2$ dimensions, Mandelstam constraints need to be satisfied[8, 9]. This can potentially be done using the virtual gluon site technique[8], so that each fermion site only has two edges and each virtual site is connected to 1)a fermionic field and two virtual sites or 2)three virtual sites. In this way, the Hamiltonian involving fermionic fields are the same as that in 1D case.

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