Towards Formalism for Quantum Chromodynamics: Loop String Hadron Formulation in SU(3) Gauge Theory

Ming Yang

Supervisor: Dr Bipasha Chakraborty

Sponsored by: CMS Summer Research Bursary

University of Cambridge

October 12, 2021

Outline

Gauge Theory

- In Quantum Mechanics, a state $|\Psi\rangle$ is indistinguishable to $e^{i\phi}|\Psi\rangle$.
- Similar for QFT
- if $\Omega(x) \in G$ for a Lie group G, and $T_a(R)$ its generators in a representation R, then under this Gauge transformation Ω
  - $\psi \to \Omega\psi$
  - matters and gauge field interact through covariant derivative, $D_\mu\psi^i = \partial_\mu\psi^i - iA^a_\mu[T^a(R)]^i_j\psi^j$
    This makes the derivative transform as the matter field $D_\mu\psi \to \Omega D_\mu\psi$
  - $A_\mu \to \Omega A_\mu\Omega^{-1} + \Omega\partial_\mu(\Omega^{-1})$

Examples

- U(1) local gauge symmetry in QED
  for $\Omega(x) = e^{i\omega(x)} \in U(1)$,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^{\mu}D_{\mu} - m)\psi \tag{1}$$

  is invariant under $A_\mu \to A_\mu - \partial_\mu \omega$, $\psi \to e^{-i\omega}\psi$

Examples

- U(1) local gauge symmetry in QED
  for $\Omega(x) = e^{i\omega(x)} \in U(1)$,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^{\mu} D_{\mu} - m)\psi \tag{1}$$

  is invariant under $A_{\mu} \to A_{\mu} - \partial_{\mu}\omega$, $\psi \to e^{-i\omega}\psi$

- SU(3) local gauge symmetry in QCD
  for $\Omega(x) \in SU(3)$,

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \Sigma_{q} \bar{\psi}^{q}_{i}(i\gamma^{\mu}(D_{\mu})_{ij} - m)\psi^{q}_{j} \tag{2}$$

  where $F^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g f^{abc} A^{b}_{\mu} A^{c}_{\nu}$
  The Lagrangian is invariant under $\psi \to \Omega(x)\psi$,
  $A_{\mu} \to \Omega(x) A_{\mu} \Omega(x)^{-1} + i\Omega(x)\partial_{\mu}\Omega(x)$.

QCD Hamiltonian

- Sign problem in Lagrangian formalism

QCD Hamiltonian

- Sign problem in Lagrangian formalism
- $$S[\phi] = \int L dt = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4 x \tag{3}$$

QCD Hamiltonian

- Sign problem in Lagrangian formalism

- Adopt Hamiltonian formalism:
  $\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$, $H = \int [\pi(x)\phi(x) - \mathcal{L}] d^3x$

QCD Hamiltonian

- Sign problem in Lagrangian formalism

- Adopt Hamiltonian formalism:
  $\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$, $H = \int [\pi(x)\phi(x) - \mathcal{L}] d^3x$
- in Weyl Gauge $A_0 = 0$,

$$H = \int d^d x [-i\bar{\psi}\gamma^i D_i \psi + m\bar{\psi}\psi + 1/2(E^2 + B^2)] \tag{4}$$

where $B_k^a = \epsilon_{klm}(\partial_l A_m^a - g/2 f^{abc} A_l^b A_m^c)$

Lattice QCD: Kogut-Susskind formulation

- Wilson line&Wilson loop
  $U[x_i, x_f; C] = \mathcal{P}exp(i\int_{x_i}^{x_f} A)$; $U[x_i, x_f; C] \to \Omega(x_i)U[x_i, x_f; C]\Omega^\dagger(x_f)$
  $W[C] = tr\mathcal{P}exp(i\oint A)$ gauge invariant
- Discretize in space:

$$H = H_I + H_M + H_E + H_\square \tag{5}$$

where

$$\begin{aligned} H_I &= -t\Sigma_{x,i}(\psi_x^\dagger U_{x,i}\psi_{x+i} + h.c.) \\ H_M &= m\Sigma_x(-)^x\psi_x^\dagger\psi_x \\ H_E &= g_e^2/2\Sigma_{x,i}E_{x,i}^2 \\ H_B &= g_m^2/2\Sigma_{x;i\neq j\in\{1,2,\dots d\}}(\square_{x,i,j}) + h.c. \end{aligned} \tag{6}$$

where the plaquettes $\square_{x,i,j} = U_{x,i}U_{x+i,j}U_{x+j,i}^\dagger U_{x,j}^\dagger$.

Schwinger Boson Formulation

- Problem with Kogut-Susskind: nonlocal nonabelian gauge freedom in Wilson line
- Idea: splitting the line into right and left part. Each part transforms as the site nearby.
  - $[E_L^a(n,i), U(n,i)] = T^a U(n,i)$; $[E_R^a(n,i), U(n,i)] = U(n,i) T^a$
  - Abelian constraints: $E_L^a E_L^a = E_R^a E_R^a$ at each link
- algebra is the same as bosonic field.
  creation/annihilation operators are similar to raising/lowering operators in SU(2)
- SU(2): 2d generators near a site $\{a_l\}_{l=L,R}$
- SU(3): 12d generators near a site $\{a_l, b_l\}_{l=L,R}$

Loop, String and Hadron

- Idea of LSH: combine sites to form locally gauge invariant states to remove nonabelian gauge
- only residue abelian gauss law
- reformulation of Hamiltonian into gauge-invariant operators
  - loop: pure gauge field
  - string: gauge field+fermionic field
  - hadron: fermionic field
  - number

LSH in SU(3)

- Complexity in Schwinger Boson representation in SU(3): reducible representation
- projected to irreducible representation with the new operators
- LSH operators
  - loop
  - string
  - hadron(mesons, baryons)
  - number operator
- gluon site: gauge loop operators

LSH operators in SU(3)-pure gauge field loop

$$
\begin{aligned}
L_{ab}^{++} &= a^{\dagger}(R)_{\alpha} b^{\dagger}(L)^{\alpha} \\
L_{ab}^{--} &= a(R)^{\alpha} b(L)_{\alpha} = (L_{ab}^{++})^{\dagger} \\
L_{ba}^{++} &= b(R)_{\alpha} a(L)^{\alpha} \\
L_{ba}^{--} &= b^{\dagger}(R)^{\alpha} a^{\dagger}(L)_{\alpha} = (L_{ba}^{++})^{\dagger} \\
L_{b}^{+-} &= b^{\dagger}(R)^{\alpha} b(L)_{\alpha} \\
L_{b}^{-+} &= b(R)_{\alpha} b^{\dagger}(L)^{\alpha} = (L_{b}^{+-})^{\dagger} \\
L_{a}^{+-} &= a^{\dagger}(R)_{\alpha} a(L)^{\alpha} \\
L_{a}^{-+} &= a(R)^{\alpha} a^{\dagger}(L)_{\alpha} = (L_{a}^{+-})^{\dagger}
\end{aligned} \tag{7}
$$

LSH operators in SU(3)-pure gauge field loop

$$\begin{aligned}
L_{ij}^{a+b+} &= a^\dagger(i) \cdot b^\dagger(j) \\
L_{ij}^{a-b-} &= a(i) \cdot b(j) \\
L_{ij}^{a+a-} &= a^\dagger(i) \cdot a(j) \\
L_{ij}^{a-a+} &= a(i) \cdot a^\dagger(j)(L_{ij}^{a+a-})^\dagger \\
L_{ij}^{b+b-} &= b^\dagger(i) \cdot b(j) \\
L_{ij}^{b-b+} &= b(i) \cdot b^\dagger(j) = (L_{ij}^{b+b-})^\dagger
\end{aligned} \tag{8}$$

$$\begin{aligned}
A_{ijk}^\dagger &= \epsilon^{\alpha\beta\gamma} a_\alpha^\dagger(i) a_\beta^\dagger(j) a_\gamma^\dagger(k) \\
A_{ijk} &= \epsilon_{\alpha\beta\gamma} a^\alpha(i) a^\beta(j) a^\gamma(k) \\
B_{ijk}^\dagger &= \epsilon_{\alpha\beta\gamma} b^{\dagger\alpha}(i) b^{\dagger\beta}(j) b^{\dagger\gamma}(k) \\
B_{ijk}^\dagger &= \epsilon^{\alpha\beta\gamma} b_\alpha(i) b_\beta(j) b_\gamma(k)
\end{aligned} \tag{9}$$

LSH operators in SU(3)-string operators

$$
\begin{aligned}
S_{in}^{b,3+} &= b(R)_\alpha \psi^{\dagger\alpha} \\
S_{in}^{b,3-} &= b^\dagger(R)^\alpha \psi_\alpha = (S_{in}^{b,3+})^\dagger \\
S_{in}^{a,3-} &= a^\dagger(R)_\alpha \psi^\alpha \\
S_{in}^{a,3+} &= a(R)^\alpha \psi^{\dagger\alpha} = (S_{in}^{a,3-})^\dagger \\
S_{in}^{b,\bar{3}+} &= b^\dagger(R)^\alpha \psi_\alpha^\dagger \\
S_{in}^{b,\bar{3}-} &= b(R)_\alpha \psi^\alpha = (S_{in}^{b,\bar{3}+})^\dagger \\
S_{in}^{a,\bar{3}-} &= a(R)^\alpha \psi_\alpha \\
S_{in}^{a,\bar{3}+} &= a^\dagger(R)_\alpha \psi_\alpha^\dagger = (S_{in}^{a,\bar{3}-})^\dagger
\end{aligned} \tag{10}
$$

LSH operators in SU(3)-string operators

$$
\begin{aligned}
S^{b,3+}_{out} &= b(L)_\alpha \psi^{\dagger\alpha} \\
S^{b,3-}_{out} &= b^\dagger (L)^\alpha \psi_\alpha = (S^{b,3+}_{out})^\dagger \\
S^{a,3-}_{out} &= a^\dagger (L)_\alpha \psi^\alpha \\
S^{a,3+}_{out} &= a(L)^\alpha \psi^{\dagger\alpha} = (S^{a,3-}_{out})^\dagger \\
S^{b,\bar{3}+}_{out} &= b^\dagger (L)^\alpha \psi^\dagger_\alpha \\
S^{b,\bar{3}-}_{out} &= b(L)_\alpha \psi^\alpha = (S^{b,\bar{3}+}_{out})^\dagger \\
S^{a,\bar{3}-}_{out} &= a(L)^\alpha \psi_\alpha \\
S^{a,\bar{3}+}_{out} &= a^\dagger (L)_\alpha \psi^\dagger_\alpha = (S^{a,\bar{3}-}_{out})^\dagger
\end{aligned}
\tag{11}
$$

LSH operators in SU(3)-hadron operators

$$
\begin{aligned}
H^{++} &= -\frac{1}{2!}\psi^{\dagger\alpha}\psi^{\dagger}_{\beta}\eta^{\beta}_{\alpha} \\
H^{--} &= \frac{1}{2!}\psi_{\alpha}\psi^{\beta}\eta^{\alpha}_{\beta} = (H^{++})^{\dagger}
\end{aligned}
\tag{12}
$$

LSH operators in SU(3)-number operators

- gauge flux operators

$$\begin{aligned} N_L &= a^\dagger(L) \cdot a(L) \\ N_R &= a^\dagger(R) \cdot a(R) \\ M_L &= b^\dagger(L) \cdot b(L) \\ M_R &= b^\dagger(R) \cdot b(R) \end{aligned} \tag{13}$$

- quark number operator

$$\begin{aligned} N_\psi &= \psi^{\dagger a}\psi_a \\ N_{\bar{\psi}} &= \psi_a^\dagger \psi^a \end{aligned} \tag{14}$$

Further work

- finish the commutation relation between the operators
- rewrite Kogut-Susskind Hamiltonian in LSH operators
- develop quantum simulation algorithms for SU(3) LGT from this formulation

References

1. Chakraborty B., et.al., *Digital Quantum Simulation of the Schwinger Model with Topological Term via Adiabatic State Preparation*, https://arxiv.org/abs/2001.00485 2. Tong D,, *Gauge Theory*, http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html
3. Georgi H., *Lie Algebras in Particle Physics*, CRC Press
4. Tong D.,*Quantum Field Theory*, http://www.damtp.cam.ac.uk/user/tong/qft.html
5. Raychowdhury I. and Stryker J., *Loop, String, and Hadron Dynamics in SU(2) Hamiltonian Lattice Gauge Theories*, https://arxiv.org/abs/1912.06133
6. Anishetty R., et.al., *Prepotential formulation of SU(3) lattice gauge theory*, https://arxiv.org/abs/0909.2394
7. Mathur M., *Harmonic oscillator pre-potentials in SU(2) lattice gauge theory* J. Phys. A: Math. Gen. 38 10015
8. Davoudi Z., Raychowdhury I. and Shaw A., *Search for Efficient Formulations for Hamiltonian Simulation of non-Abelian Lattice Gauge Theories*, https://arxiv.org/abs/2009.11802
9. Chandrasekharan S.and Wiese U., *Quantum Link Models: A Discrete Approach to Gauge Theories*, https://arxiv.org/abs/hep-lat/9609042
10. Manton N., *The Schwinger Model and Its Axial Anomaly*, Annals of Physics **159**(1985), p220-251
11. Kogut J. and Susskind L., *Hamiltonian formulation of Wilson's lattice gauge theories*, Physical Review D **Vol. 11**(1975) 2, pp395-408
12. Muschik, C. et. al., *U(1) Wilson lattice gauge theories in digital quantum simulators*, https://arxiv.org/abs/1612.08653